ARTHUR CAYLEY.

BORN AUGUST 16TH, 1821. DIED JANUARY 26TH, 1895.

" My subject is the life of a great artist."

WHILE it is fitting that the BULLETIN of the American Mathematical Society should make mention of the passing away of professed mathematicians, the death of one so great as Arthur Cayley demands more than a slight notice. My willingness to undertake the task is due, not to any sense of fitness on my part, but to my intense admiration for his work and for his personality, and to the fact that for the last fourteen years I have been privileged to know him and experience his kindness.

The facts of his uneventful life are given by his friend, Dr. Salmon, in *Nature* for September, 1883. He was the second son of an Englishman, a merchant in Russia, and was born in England during a visit his parents paid to their home. In 1829 they returned to England to live, so Professor Cayley was English by education as well as by nationality. He entered Trinity College, Cambridge, in 1838, and in 1842 was Senior Wrangler and first Smith's Prizeman. He entered the legal profession, and practised until 1863, when he returned to Cambridge to hold the Sadlerian Professorship of Pure Mathematics; this he held until his death. He has published about 800 papers, of which the first appeared in 1841; the last may easily turn out to be the one contained in the January number of the BULLETIN, for this was written December 18th, 1894. It would have been a cause of special regret to all interested in the American Mathematical Society had the pages of the BULLETIN contained no contribution from our most distinguished member.

At the time when Cayley entered Cambridge the mathematical curriculum was very different from the present one. Text-books were few and soon exhausted; but in this there was the advantage that so much the sooner was the student thrown into direct contact with the works of the great mathematicians. As regards the majority of students, it may have been that they had nothing to draw with, and the well was deep; but for one like Cayley, to whom the most abstract language of mathematical analysis was as his mother-tongue, the well was as a spring bubbling up. Nor was he alone in this direct communication with the fountain-head; his contemporaries in Cambridge were such men as Leslie

Ellis, Stokes, Adams, Thomson. The time too was favorable. Whether or no one accepts Bacon's dictum, that knowledge after it has been systematized is less likely to increase than before, it is plain that periods of growth, creative periods, alternate with quieter times of reflection and systematization, critical periods. Now that the lines of mathematical research are so many and various, this sharp alternation is less noticeable; but the time we are considering was distinctly a creative period. Analytical methods had just succeeded in displacing the fluxional and geometrical methods which had held undisputed sway in England from the time of Newton, cutting off English mathematicians from intercourse with the Continent of Europe, and exercising a blighting effect that is frequently referred to; see for instance Hankel's Entwickelung der Mathematik:—"Die Engländer, bei denen bis vor Kurzem aus Pietät gegen ihren grossen Landsmann Newton die Mathematik völlig still stand." This change had been brought about by the efforts of Peacock, Herschel, and Babbage, and was an accomplished fact by 1830. The influence that made the change possible was felt as a general quickening all along the line; among other manifestations of it we may note the founding of the Cambridge Philosophical Society in 1819, and the establishment in 1837 of the *Cambridge Mathematical Journal*.

As regards the mathematical literature that was available, the works of the great French analysts were already classics; Gauss, Poncelet, Möbius, Chasles, Steiner, Plücker, Jacobi, were in the height of their activity; Abel's life had actually ended, but the influence of his too-lately-recognized genius was just beginning to be felt. During the twenty years immediately preceding Cayley's graduation there had appeared (among other works) the *Propriétés projectives* of Poncelet, the *Barycentrische Calcul* of Möbius, the *Systematische Entwickelung* of Steiner, Jacobi's *Fundamenta Nova*, Chasles' *Aperçu Historique*, the *Analytisch-geometrische Entwickelungen*, the *System der analytischen Geometrie*, and the *Theorie der algebraischen Curven*, of Plücker. Apart from any conjectures as to what Cayley read first, we know from his own words, in papers published while he was yet an undergraduate and shortly after, that before 1843 he was familiar with the works of Laplace and with a wide range of memoir literature in the journals of Liouville and Crelle.

Some of the most striking of Professor Cayley's earlier papers deal with the subject of Elliptic Functions, which had been treated in two very different ways by Abel and Jacobi. For the work of both these mathematicians Cayley had a great admiration, which as regards Abel, who died in 1829, at the age of 26, was intensified by keen interest in the man

himself. No one who attended Professor Cayley's lectures on Abel's Theorem, delivered in Cambridge in 1881, could fail to be struck by the note of personal regret with which Abel was referred to. Abel's great paper, neglected by the French Academy of Sciences for fourteen years, was finally published in 1841, and probably this had brought vividly home to Cayley the pathetic story of the great Norwegian. The functions discussed by Jacobi are singly-infinite products, those of Abel's papers are doubly-infinite products; Cauchy had shown that the properties of the Elliptic Functions could be deduced from their definitions by means of Jacobi's singly-infinite products; Cayley showed that Abel's doubly-infinite products define functions which also satisfy the equations that define Jacobi's Elliptic Functions. In the note to the reissue of these papers (Nos. 24 and 25 in the Collected Papers) Cayley remarks that the need for this investigation existed at the time, though it has since been superseded by the work of Weierstrass.

Many of the early papers exhibit a wide acquaintance with the works of various writers; perhaps the one whose influence is most plainly to be discerned then and at a later time is Plücker. But this influence seems to have been exhibited more as an impulse starting Cayley's genius along certain lines, and providing him with preliminary subject-matter, than in any modifying or formative effect. In their conviction of the inherent identity of algebraic and geometric operations Plücker and Cayley were at one; in the marvellous richness and plenitude of their ideas, their creative power, they resembled one another; but the elaborated carefulness of Cayley's work, in lectures as in memoirs, contrasts with the hasty intuitiveness of Plücker, to which Clebsch bears witness; Cayley's extensive acquaintance with mathematical literature, and the breadth of his interests, with Plücker's very small knowledge of what his contemporaries were doing or had done; and, for a most striking difference, Plücker's style is so repulsive that probably very few know him at first hand.

Plücker's works on Plane Geometry, though published at intervals from 1828–1839, were evidently not well known to Cayley until after 1843, though he refers to special papers; it is plain that Chasles' Aperçu Historique was first known to him; for in No. 5, a classical paper of three quarto pages, originally published in 1843, he ascribes to Chasles the demonstration of Pascal's Theorem by means of the theorem that every cubic through eight of the intersections of two cubics passes through the ninth; this is however to be found in Plücker, Entwickelungen, I., pp. 266–7, antedating Chasles by nine years. Cayley reproduces the proof, with the

remark that it ought to be generally known, and after proving the nine-points theorem, which had simply been enunciated by Chasles, he passes on to give the theorem known by his name, which limits the number of the points determining a curve of order r that can be chosen common to two curves of orders m and n. It is a matter of some historical interest that Clebsch, writing in December, 1871, speaking of this memoir, says that with it the algebraic side of this investigation can be regarded as finally closed. It was, however, reopened in 1881 (*Mathematische Annalen*, 1886) by Bacharach, who points out a limitation which, though referring to a group of special cases, is yet of sufficient importance to be incorporated in the theorem. The theorem on the intersections of curves is fundamental in the fabric raised up by Brill, Nöther, and others.

Accepting the broad division of mathematical thought (given in the notice of Clebsch, *Mathematische Annalen*, vol. 7) into two main lines,—the one concerning itself with the precise defining and justifying of the fundamental conceptions of the science, manifesting itself in investigations in the Theory of Numbers and the general Theory of Functions, connecting itself in the first instance with the names of Gauss and Dirichlet; the other accepting a certain small number of fundamental conceptions and working from these, thus producing modern algebraic and geometrical investigations, and to be traced to the influence of Jacobi and Plücker,—there is no difficulty in assigning Professor Cayley to the latter division. He himself stated that he had not been able to feel any special sympathy with Riemann's ideas, and this is plainly indicated in his address as President of the British Association for the Advancement of Science, 1883, where after a mere passing reference to the general subject he left it and discussed *con amore* the special classes of Elliptic and Abelian Functions to which so much of his thought had been devoted. I never heard him assign any reason for this feeling, nor can I find in his papers anything to explain it; his interests were certainly wide enough to embrace this field, and it is well known with what open-minded warmth he welcomed new thoughts. Probably the explanation will be given by some of his intimate friends.

In speaking of Cayley's work, and attempting to give a reason for ranking him first among the mathematicians of the age, it is easiest to speak of his work on Invariants, for the reason that here we can point to a single theory, by universal consent due to Cayley, which as a matter of general knowledge has dominated one half of mathematics ever since its first acceptance. Every entity is endowed with properties inherent in itself, which cannot be affected by any change of base

on our part, so long as this change of base is no more than a change of what we assume as fixed, or of our standards of measurement, or (speaking for the sake of definiteness of such entities only as are contained in a plane) of the angle under which we view the entity. In its most simple geometrical form this is identical with the theory of conical projection, and the geometrical permanence here exhibited was well known; in the algebraic form the theory is that quantics homogeneous in any number of variables have properties depending on certain allied expressions that are not affected by linear transformations. Some of these allied expressions had of necessity been noticed, since it is impossible even to solve an equation with literal coefficients without coming upon them; the great merit of Cayley's work is the recognition of this permanence of relation, this Invariance, as an attribute to be recognized for its own sake, and investigated *a priori*. The importance in result is twofold: it gives to algebraic forms a substantive existence, and it leads of necessity to the conception of groups of transformations. So thoroughly has this theory permeated all mathematical work, that it is hard to realize that the jubilee of the discovery occurs only this year, Cayley's original paper, "On the Theory of Linear Transformations," having been published in 1845.

Another conception that can be mentioned as entirely due to Cayley is that of the Absolute. The existence of the circular points in a plane was known; the fact that the angle between two lines was really a measure of their relation to the circular points had been shown by Laguerre. It is easy to show that in a plane the analytical expression of any property depending on exact measurements really expresses a relation of the figure to the circular points, and that the relations expressed are simply such relations of position as can be held by the figure to a conic: and that similarly in space of three dimensions the relations are such as can be held to a quadric surface. Cayley's conception is then that instead of discussing the metric properties of a system directly, the relations of position with respect to a quadric surface may be considered, the metric properties being deduced by properly choosing the quadric. To quote his own words, from the Sixth Memoir upon Quantics, 1858,—"The theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connection with another figure, the Absolute." This theory of Cayley's deals primarily with Euclidean space, giving a more general principle for dealing with measurements. If however the fundamental quadric be differently chosen, we are led to the theories of non-Euclidean space. Professor Cayley's concern was not with these; his

object was to obtain a general conception of the nature of measurement in space, and thus to subject metric properties to descriptive treatment. The effect of this idea on the development of mathematical thought can be well seen, to name only one instance, in Professor Klein's "Review of Recent Researches in Geometry," especially in § 2. (See vol. 2 of this BULLETIN.)

In other matters it is not so easy to give an account of what Cayley has done; it is hardly possible to point out a region, complete in itself, as his; his work is interwoven with that of his time, and not until the history of the development of mathematics during the present century comes to be written by a competent hand will it be possible to form a just estimate of his overwhelming importance. Apart from his minute and extensive series of investigations in solid geometry, algebra, and questions of analysis, there are numerous detached papers that have put a question in its true light, and so served as starting-points for the researches of other mathematicians. For instance, Cayley by his investigations in the theory of Higher Singularities prepared the way for more recent discussions of the question. (See Nos. 343, 374, published in 1864 and 1866.) Plücker, using the dual generation of the curve by a moving point and a moving line, had arrived at the conclusions that a point at which two branches of a curve have contact of order g must be regarded as $g+1$ double points, and that there are also $g+1$ double tangents; and that a cusp of the first species is to be counted as one and a half nodes, a cusp of the second species as two and a half. While this result, as a numerical one, is correct to a certain extent, yet as a geometrical view it is essentially incorrect, for it assumes that cusps of different species are different in nature. Cayley showed that the apparent difference is due to the fact that the cusp of the second species is a cusp with an adjacent node; and that it contains also a double tangent, in addition to the inflexional tangent recognized by Plücker; he shows in general what is the nature of the branches that may occur, and explains how the number of ordinary cusps and nodes included in any singularity can be determined; he considers also the determination of the inflexional and double tangents. He does not consider the grouping of the double points into multiple points, nor the possibility of separating the component multiple points from one another; for present use these are important questions, and the process of Cayley, proceeding by expansions, is to a great extent replaced by the successive reduction by means of quadratic transformations, a process which reveals not only the number but also the inner arrangement of the elements of the singularity. But however the work be arranged, to Professor Cayley is due the fundamental

theorem that any singularity that can occur on an algebraic curve can be fully accounted for by a definite and determinable number of cusps and nodes, inflexional and double tangents.

Moreover, to Professor Cayley we owe a debt not only as an explorer, but as one who followed out of set purpose the tracks often so faintly indicated by those that first passed that way. Where he found a mere mountain-path he left a broad road, steep it may be, but plainly marked out, and with no hidden pitfalls or lurking dangers. While it is his creative power that excites our admiration, it is this determination that others shall be enabled to follow that endears him to us. This he has done consistently for his own work as well as for that of others, and it is partly due to this that so large a proportion of the results and processes of his earlier papers have become the commonplaces of text-books. To this however another cause has powerfully contributed. With the single exception already noticed, Cayley was emphatically in touch with the mathematics of the day; his omnivorous reading, his rapid assimilation of new thoughts, his readiness to believe in others, his frank and genial response to any appeal for help, all combined to make him the ideal head of the mathematical world.

All this the historian of a century hence may know; it is written in the unintentional references of his contemporaries as well as in their deliberate tributes. But any sketch of Professor Cayley is self-condemned if it leaves out of account the childlike purity and simplicity of his nature, the entire freedom from the professional touchiness on the score of priority to which mathematicians are as liable as other men. He was ever ready to say what he was working at, to indicate the lines of thought, to state what difficulties he was encountering. It is not every mathematician that will lecture to a class of specialists on the incomplete investigation of the night before, and end up with the remark, obviously genuine, "Perhaps some of you may find this out before I do." To this engaging sweetness of his nature must be ascribed also the urbanity of manner which, as one of its many manifestations, would not allow him to leave unacknowledged the slightest mathematical paper sent to him. His little note of thanks would frequently contain a few words of comment, just enough to show that he had made time to glance over the paper. Anecdotes illustrating his absorption in his work, his disregard for appearances, are well known to all that knew him, and will doubtless be forthcoming in the reminiscences of his contemporaries; but his greatness and his simplicity cannot be enshrined in anecdotes.

My own acquaintance with Professor Cayley dates back to

1880, the first of the four years in which I attended his lectures. His subjects in the different years were Modern Algebra, the Abelian Functions, the Theory of Numbers, the Theory of Substitutions, the Theory of Seminvariants. To my great regret I never had the opportunity of hearing him lecture on Geometry. His lectures differed strikingly from his memoirs in that the subject was presented in less synthetic style. It was a recognized fact that he lectured on what he himself was working out at the time, and consequently his class was privileged to obtain some insight into the workings of his mind. The necessary connection of the different ideas presented was perhaps not always obvious at the time—at any rate to some that were not very familiar with what had already been done in that line; but it is my belief that Cayley had the power of communicating to his hearers, for the instant at any rate, something of his own clear vision of the essential nature of the truth he was expounding.

He has been spoken of as the greatest living master of algebra. In an interesting estimate of his genius and personality, contained in the *Bulletin des Sciences Mathématiques* for 1893, Darboux quotes and endorses the opinion of Bertrand, who compares him with Euler, instancing as one reason among many his extraordinary power and certainty in analysis. Some, passing lightly over the geometrical side of his investigations, and perhaps dazzled by his supremacy over the rest of the world in analysis, appear inclined to consider him only under this aspect. Professor Klein, in his Evanston Lectures, dividing mathematicians into logicians, formalists, and intuitionists, classes him as a formalist, a term not altogether pleasing, inasmuch as it conveys a shade of reproach, but which is immediately explained by the speaker. That Cayley did "excel in the skilful formal treatment of a given question, in devising for it an algorithm," is beyond question; but I doubt whether there will be equally general agreement in the statement that he excelled "mainly" in this. It has always seemed to me that while Cayley's processes were algebraic, since the language of algebra was simpler to him than the ordinary language of words, the color of his thought was essentially geometrical. But algebraic and geometric operations are so absolutely identical in his work, that the passages which convey to one mind the idea of a geometrical substratum may very well convey to another the idea that nothing existed for Cayley but the analysis. It is curious sometimes to note how little he realizes what a barrier extensive analytical operations are to the progress of many students. I have lying before me a note, dated December 1894, in which he urges his view that a complete knowledge of in-

variants and covariants of ternary forms ought to be presupposed in the teaching of Higher Plane Curves.

Cayley's literary style is simplicity and directness itself, in marked contrast to that of Sylvester, where, according to Cayley, "there is not unfrequently an adornment or enthusiasm of language which one admires or is amused with." As regards the mathematical style, the marshalling of the subject-matter, it is distinguished by its cumulative force. His memoirs are absolutely convincing, though they may not be understood at the first or second reading. His writing has not the seductive literary appearance of extreme simplicity, concealing unsuspected depths, now of difficulty, now of allusion, so characteristic of Dr. Salmon; and very contradictory opinions have been expressed as to the effect of his style. Dr. Salmon thinks that students in general find Cayley's papers hard to understand; Darboux, on the contrary, speaks of "ces productions, si variées et par le sujet et par la méthode, dont la lecture est facile, où rien n'est laissé dans l'ombre, pas même les sujets de nouveaux travaux et les points à approfondir." These opinions, contradictory as they seem, are however easily reconciled. No indolence is admissible over a paper of Cayley's; there is no going rapidly over it, with the hope of getting a superficial idea, leaving difficulties to be considered at a more convenient season; it must be all or nothing. But when the reader is prepared to work through the subject with the author, following closely in his steps and trusting to his guidance, he will find that there are no real difficulties beyond the great preliminary one of the subject itself.

Moreover, Cayley's memoirs (on pure mathematics, at any rate) are almost invariably interesting and inspiring. They abound in suggestive remarks, indications of interconnection, historical references, sketches of the processes and conclusions of his forerunners. In reading them one feels that but for one's own deficiencies it would be easily possible by their means to penetrate into the very heart of the subject. For Cayley's supreme power was happily that of an investigator rather than of a lecturer; and in the volumes now being issued by his university we have as much of himself as a great man can leave behind him.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
February 18, 1895.

THE THEORY OF FUNCTIONS.

Theory of Functions of a Complex Variable. By A. R. Forsyth, Sc.D., F.R.S. Cambridge, University Press, 1893. 8vo, xxii + 682 pp.

The year 1893 was marked by the appearance of four important treatises on the theory of functions. The second volume of Picard's *Traité d'Analyse** presents, as no other one book has ever yet done, the ideas that originated with Riemann and have been developed by him and his followers. The first volume of the new edition of Jordan's *Cours d'Analyse*† begins, to be sure, with the elements of the infinitesimal calculus, but it is only after such a treatment of the foundation on which analysis rests, enriched by the methods and dominated by the spirit of what is commonly known as the modern theory of functions of a real variable, that Cauchy's theory of functions can be developed in a manner that will be adjudged rigorous by the standards of this age.

On the other hand, Harkness and Morley's treatise,‡ like the work before us, is not confined to any one of the three theories due respectively to Cauchy, to Weierstrass, and to Riemann; and these are the first works to appear on the theory of functions in which all three theories are treated. While they have a common object, namely, to introduce the student to the whole field of the theory of functions, the manner in which this plan is carried out is quite different. The former writers feel keenly the advances that have been made in recent times in accuracy and rigor in analysis. They know that this work has as yet found scarcely any place in the text-books, and that a knowledge of it cannot be assumed on the part of the student who has just finished his differential and integral calculus. They, therefore, devote some introductory chapters to it and emphasize it throughout the subsequent developments of their subject. Our author, on the other hand, gives little heed to such matters. He takes the ground that the student who is interested in these questions can find them treated well enough elsewhere; and he does not regard it as part of his task to point out to the student such difficulties and thus to foster in him a critical spirit, nor does he shape his proofs, as Picard has done so successfully, so as to avoid these difficulties, or, if that is not possible, to meet them squarely. This, from the author's point of view.

* Reviewed in the Bulletin, vol. 3, p. 39.
† Reviewed in the Bulletin, vol. 3, p. 135.
‡ Reviewed in the Bulletin, vol. 3, p. 155.

But we cannot stop with pointing merely to the loose form in which theorems are often stated and proofs given; it only too often happens that the ideas on which the proofs rest are lacking in rigor, or that important matters are overlooked.

It is thus easy to understand that Cauchy's theory appears simpler to the author than Weierstrass's. For the difficulties in the former are hidden in the elements of the older analysis, supposed known; while those of the latter centre in theorems regarding series, of which, though they are equally elementary—in some respects even more so—elementary arithmetical proofs are not generally known. We would not be understood as objecting to the author's use of Cauchy's theory. It seems to us an excellent feature of this treatise that the author has always aimed to use that method which is best adapted to the proof of the proposition in hand; and certainly Cauchy's methods, supposing them to have been once rigorously established, often lead more directly to the desired result than Weierstrass's. Riemann's theory of the algebraic functions and their integrals is set forth. But the possibility of basing the whole theory of functions on the logarithmic potential function, which is the leading idea in Riemann's theory (and which is worked out in detail, for example, in Picard's treatise), is not clearly brought out.

In connection with this treatment of the theory of functions, however, a further point deserves attention. While the student gets, by the alternation in the use of Cauchy's, of Weierstrass's, and of Riemann's methods, a broader view of the subject as a whole, he fails to get a due appreciation of the homogeneity of each method by itself. This difficulty could be met, in great measure, by brief sketches of the lines along which important results, obtained by use of one method, would be established by the other method, such sketches to be accompanied by full references regarding details. It may be said that this is a matter of interest only to mature students. But it is for such that the whole treatise will be of greatest value. It gives them on its pages a "consecutive account of what may fairly be deemed the principal branches of the whole subject" of the general theory of functions of a complex variable, and the full lists of bibliographical references guide them in making a systematic study of those special subjects on which they wish to inform themselves in detail. The service that the author has rendered to them in compiling these lists is great and will be deeply appreciated.

One important method we miss altogether. Transformations are nowhere treated, the linear transformation in its relation to automorphic functions alone excepted. Thus, though considerable space is devoted to algebraic functions

and their integrals, the rational transformations of one algebraic configuration (Gebilde) into another, together with the invariants of such transformation—the moduli (Moduln) of the Riemann's surface—and the transformations of the periods of the integrals, in particular, of the elliptic integrals, are only mentioned in passing.

An excellent feature of the book, and a feature distinctly English, is the frequent introduction of problems, illustrative of the principles treated, for the reader to solve. It is a mistake too frequently made, alike by teachers and students, to believe that one understands a theorem because one can prove it. The real test is whether one can apply it, and for this test examples are useful.

A glossary of technical terms, an index of authors quoted, and a general index—the latter covering over 18 pages—are appended.

Chapter I is entitled General Introduction. In § 1, a brief statement is made regarding the relation of the algebra of ordinary complex quantities, $a + bi$, to other multiple algebras; but it is impossible to make this relation clear in half a page. The geometric representation of the complex variable by a point in the plane or on the sphere of stereographic projection is explained, but no reason is given for regarding the infinite region of the plane as a point, it being apparently sufficient that, in the projection on the sphere, one point of the latter remains unprovided for.—If u, v be any functions whatever of x, y, then $u + vi$ will be a function of $z = x + yi$ in the sense that, when z is given, w is determined. In the case of the ordinary functions (rational functions, functions defined by convergent power series, etc.), z being taken as the independent variable, u and v are restricted,—between their partial derivatives with regard to x and y there exist, for example, certain relations. The author regards this restriction as seated in the fact that the complex variable $z = x + yi$ has been operated on as an "irresoluble entity," and, generalizing, regards a function of z as generated by an operation on z as a whole, not on x, y separately. This idea is not a new one. It is, however, open to attack, for it is not clear that the above requirement sifts out from the general class of functions, $u + vi$, the special class that is desired, since the kind of operation, here supposed arithmetical, to be performed on z has not been restricted. For example, the projection of the z plane on the w plane is a geometric operation that affects z as a whole and that defines w geometrically as a function of z; and it is conceivable that a convergent arithmetic process exist which, operating on z as an "irresoluble entity," would yield the same function.—On p. 15, the definition, "a point in the plane, at which two or more

branches of a multiform function assume the same value, is called a branch-point," is inaccurate. Thus $z = 0$ is not a branch-point of the function w, where $w^3 = z^2(1 - z)$. An accurate definition is given on p. 162. The choice of the word *uniform* for single-valued does not seem fortunate; still less so, the rendering of *ausserwesentlich singuläre Stelle* by *accidental singularity*. The author always speaks of *rational algebraic* functions and *rational integral algebraic* functions; the reason for inserting the word *algebraic* in such cases is not clear.

In Chapter II, the integral of a complex function is defined, its existence being assumed, and elementary theorems regarding such integrals are established. On p. 20, the theorem is stated: "When a function is expressed in the form of a series, which converges uniformly and unconditionally, the integral of the function along any path of finite length is the sum of the integrals of the terms of the series along the same path, provided that path lies within the circle of convergence of the series." The requirement of unconditional convergence is irrelevant. In § 17, Cauchy's fundamental theorem: "If a function $f(z)$ be holomorphic throughout any region of the z plane, then the integral $\int f(z)dz$, taken round the whole boundary of that region, is zero," is proved by double integration, no mention being made of other proofs. In the statement of the theorem of § 20, the words "which is not a zero of $f(z)$" should be omitted. One is at a loss to see why, in § 21, so careful a proof is given that differentiation under the sign of integration is allowable. Usually, such matters are not touched upon. The chapter closes with some good examples in illustration of the preceding principles.

Chapter III begins with a careful proof of Cauchy's theorem regarding the expansion of a function into a power series. Laurent's theorem (p. 47) is stated as follows: "A function which is holomorphic in a part of the plane bounded by two concentric circles with centre a and finite radii can be expanded in the form of a double series of integral powers, positive and negative, of $z - a$, the series converging uniformly and unconditionally in the part of the plane between the circles." The error here made regarding uniform convergence occurs repeatedly throughout the book. Consider, for example, the series

$$1 + z + z^2 + \ldots$$

Let r be any positive real number less than 1. Then this series converges uniformly within a circle of radius r about the point $z = 0$. r can be taken as near to 1 as desired and the convergence will still be uniform. Nevertheless, the series

does not converge uniformly within the unit circle about $z = 0$, for the remainder after n terms is

$$\frac{z^n}{1 - z},$$

and, after a positive quantity ϵ has been chosen at pleasure, it is evidently *not* possible to find a positive integer ν such that, for all values of $n > \nu$, the absolute value of this remainder will lie below ϵ, no matter how z be now chosen within the unit circle about $z = 0$. A power series always converges absolutely within its circle of convergence, but it may not converge uniformly. If it does, however, then it converges uniformly *within and on* this circle. In particular, a power series can converge *absolutely* for all points of the plane, but it cannot converge *uniformly* for this region, unless it be a constant. Th[illegible], incorrect statements occur on pp. 76, 90, 137, 142, and [illegible]ewhere. In fact, the author often requires of a series whose terms are not even functions of a variable that it converge uniformly! Cf. pp. 76, 83, 85. In the definition of uniform convergence, given on p. 127, the essential point is not stated, and we almost fear that it has escaped the author's attention.—§§ 32, 33 treat of the singularities of a single-valued function. It is tacitly assumed that these are *isolated* poles or essential singularities. In § 33, the behavior of a single-valued function in the neighborhood of an essential singularity is considered. The investigation is based on the theorem that, if a function $f(z)$ be holomorphic at all points in the neighborhood of $z = a$, then either (1) it is holomorphic at $z = a$ also; or (2) there exists a series of values $z_1, z_2, \ldots$, $\lim z_i = a$ for $i = \infty$, such that $\lim f(z_i) = \infty$ for $i = \infty$. This theorem is not clearly established by the considerations of these paragraphs; it is, however, evident that the author has these relations clearly in mind. Then: "Further, a uniform function must be capable of assuming any value C at an essential singularity." Now the function is not defined by the power series in the singular point, and whether we say, with Continental mathematicians, that it shall not be defined at all at this point or, with the author, that it shall be defined to have all values is, perhaps, a matter of taste. The important fact is this: let C be any real or complex quantity, let a circle of arbitrary radius be drawn about the essential singularity, and let ϵ be an arbitrarily chosen positive quantity. Then it is always possible to find points within the circle (exclusive of the centre) in which the difference between C and the value of the function is, in absolute value, less than ϵ. The function may or may not attain the value C. Thus the function

$$1 + z + z^2/2\,! + \ldots$$

never attains the value 0. This fact is not brought out. Cf. also p. 64, Cors. I, II, and p. 219. The chapter ends with a clear account of Weierstrass's method of continuation of an analytic function and Schwarz's continuation by the principle of symmetry.

Chapter IV discusses single-valued functions, in particular, rational functions. Theorem I reads: "A function which is constant throughout any region of the plane not infinitesimal in area, or which is constant along any line not infinitesimal in length, is constant throughout its region of continuity"; and at the end of the proof: "It should be noted that, if in the first case the area be so infinitesimally small and in the second the line so infinitesimally short that consecutive points cannot be taken, then the values at a of the derivatives cannot be proved to be zero, and the theorem cannot then be inferred." Here is the doctrine of infinitesimals so popular in earlier times. That doctrine regards an infinitesimal not as a variable, but as a constant, an "infinitely small constant," having geometric existence (e.g., an "element" of arc, area, etc.); and hence two fixed points on a line that do not coincide need not be situated a definite distance apart from each other. But this does not correspond to our intuition of a line, and geometric intuition was formerly the foundation on which arithmetic rested. In recent times, arithmetic has been developed independently of geometry, and it now stands on the firmest foundation known to mathematicians. In its turn, it has thrown light on the nature of our geometric axioms; and not the least important result that it has achieved is the demonstration of the impossibility of such a doctrine of infinitesimals as the one the author holds. Many other passages may be cited in this connection; of such the following are a few: p. 29, end; p. 45, near end; p. 201, middle; p. 227, Cor. II, note. On pp. 60, 61, we read: "Further, in any finite area of the region of continuity of a function that is subject to variation, there can be at most only a finite number of its zeros, when no point of the boundary of the area is infinitesimally near an essential singularity." The words "infinitesimally near" should be omitted; for, if the region contains an infinite number of zeros, there will be at least one essential singularity on the boundary. In the proof that follows, the inference that the zeros would fill out a continuous line or area is incorrect. Cf. also pp. 206, 218. In the subsequent pages of the chapter, the fundamental theorems relating to rational functions are established.

In Chapters V, VI, VII, the investigations of Weierstrass and Mittag-Leffler on single-valued transcendental functions, together with important examples given by Tannery and Poincaré, are treated at great length (pp. 74–148). In the

last three lines of § 50: "Further, $z=\infty$ is an essential singularity of the function; for it is an essential singularity of each of the factors on account of the exponential element in the factor," the reason assigned is wrong, as is shown by the following example:

$$\prod_{0}^{\infty} p_n, \quad \text{where } p_0 = e^z, \quad p_n = e^{\frac{1}{(n+1)z} - \frac{1}{n^2}}, \quad n > 0.$$

On p. 77, l. 17, "... and taking ϵ...," read k for ϵ. On p. 79, the author wishes to prove that an infinite product, $G(z)$, has an ordinary zero in a point in which one of its factors has an ordinary zero. For this purpose he says (l. 8): "Since the product $G(z)$ converges unconditionally, no product constructed from its factors E, say from all but one of them, can be infinite." The property in question does not follow from the unconditional convergence, nor even from the uniform convergence. It follows from the latter, combined with the fact that the individual factors are analytic functions throughout a common region.

In § 82, Weierstrass's theorem regarding the expression of a series of analytic functions $\Sigma f_i(z)$ as a series of positive and negative powers of z is stated inaccurately, and the proof is confused. Weierstrass long ago published* his proof of this theorem, which is a model of clearness and rigor, and which the author would have done well to translate literally. The further theorem that such a series $\Sigma f_i(z)$ can be differentiated term by term is omitted. When such series arise, they are so differentiated without further ceremony (v. § 124). In § 57, Weierstrass's σ-function is defined. The main results obtained in these chapters consist in the establishment of the various forms in which, as Weierstrass and Mittag-Leffler have shown, single-valued functions, especially those with a single essential singularity, can be expressed when their zeros and poles are known. Good examples are given. The investigations of Cantor on clusters of points find no place in the book. Such clusters cannot be avoided in Mittag-Leffler's theorems, so the author skilfully restricts his treatment to the case where the derived cluster is the circumference of a circle (p. 117). In § 87, Poincaré's examples of functions with lacunary spaces are considered, but the reasoning that the author gives is wrong. It does not follow from the fact that the terms of an infinite series

$$\sum_{1}^{\infty} \frac{A_i}{z - b_i}$$

* *Monatsber. d. Berl. Akad.*, 1880.

have poles clustering about a point b, that this point will be an essential singularity for the function defined by the series, as is here assumed. Consider, for example, the series

$$-\frac{1}{z}+\frac{z}{z^2+1}+\sum_{1}^{\infty}\left(\frac{z}{z^2+\frac{1}{n+1}}-\frac{z}{z^2+\frac{1}{n}}\right)$$

which has the value 0 for all values of z for which its terms are defined, and which can be thrown at once into the above form. Its terms have poles clustering about the point $z=0$, but this is not a singularity of the function. The restriction that Poincaré makes on the coefficients A is essential. Cf. Poincaré's memoir in the *Amer. Jour. of Math.*, vol. 14.—The results obtained in the memoirs under consideration are, indeed, given in these chapters, but the careful proofs of the original are not preserved.

With Chapter VIII, the treatment of multiple-valued functions begins. This chapter is more a description than a careful treatment of the general properties of such functions. Branch-points are surrounded with loops, and the Riemann's surface is not introduced till later. The transformations that lead to the separation of the branches of an algebraic function at a singular (multiple) point are considered, but no proof is given that a finite number of applications of such transformations will eventually effect the desired separation; indeed, no proof is given that an algebraic function is analytic. Surely we could have spared some of the 74 pages of Chapters V–VII better than to do without these theorems.—The case treated by Clebsch and Gordan in their *Abelsche Functionen* of a curve with only double points and cusps (the corresponding Riemann's surface having, accordingly, only simple branch-points) is designated by the author (§ 98) as "of considerable importance, though limited in character." From the important point of view that two algebraic configurations (Gebilde) are equivalent when one can be transformed rationally into the other, there is here no limitation whatever.

Chapters IX–XIII are devoted to periodic functions. Integrals with moduli of periodicity are discussed, and Hermite's method is described. Functions defined by definite integrals are considered, and the double-circuit integrals introduced by Jordan and Pochhammer are referred to, but not discussed. Jacobi's arithmetic proof that a single-valued function cannot have more than two periods is given, but no geometric proof, like that given, for example, in Briot and Bouquet's *Fonctions elliptiques*, 2d ed., bk. IV, ch. III, is mentioned. The treatment of simply periodic and doubly periodic functions, in Chapter X, would have been rendered clearer if it had been illus-

trated by the corresponding conformal representations. The transformations $Z = e^{\frac{2\pi i}{\omega} z}$ and $Z = \tan \frac{\pi z}{\omega}$ are studied arithmetically, and these functions are used as prime functions in terms of which simply periodic functions are expressed. The theorem near the top of p. 215 is not stated accurately. The theorem of § 114 is incorrect; consider $e^{z^2} \tan z$. The theorem that no rational integral function can be simply periodic is omitted, though the corresponding theorem for double periodicity is stated and proved. Next, the fundamental theorems regarding doubly periodic functions are developed, and the general differential equations of Jacobi's and of Weierstrass's elliptic functions are deduced analytically from these theorems. —Doubly-Periodic Functions of the Second Order is the title of Chapter XI. Weierstrass's σ-, ζ-, and $\wp$-functions are introduced. Throughout these chapters, the general line of thought is presented clearly, but details are often omitted which the student cannot supply, as, for example, in § 124, already referred to.—Chapter XII is on Pseudo-Periodic Functions, of which θ and σ are examples, and Hermite's application to the solution of Lamé's equation (§ 140) is discussed.—In Chapter XIII, Weierstrass's theorem relating to analytic functions with an algebraic addition-theorem is taken up. Phragmén's proof is referred to, but a different proof is given by the author. This latter proof is, however, incomplete in some essential points. The assumption that forms the starting-point in Weierstrass's theorem is that, between three convergent power series, $P_1(u - u_0)$, $P_2(v - v_0)$, $P_3(w - \overline{u_0 + v_0})$, elements of the same analytic function, $\phi(u)$, there exist, when w is put equal to $u + v$, an algebraic relation,

$$G\{P_1(u - u_0),\ P_2(v - v_0),\ P_3(u + v - \overline{u_0 + v_0})\} \equiv 0;$$

and the existence of the same algebraic relation between $\phi(u)$, $\phi(v)$, $\phi(u + v)$ must first be established. All this, the author omits. He further assumes that if $\phi(u)$ is defined for u and for v, then it is also defined for $u + v$. But this is not true, even in general, of all analytic functions. When these results have been established, however, the further proof along the lines that Phragmén follows is much simpler than the one the author gives.

In Chapter XIV, entitled Connection of Surfaces, the author gives a clear and simple account of this subject. Only bifacial surfaces are considered. Systems of irreducible circuits are treated, and the way is prepared for Chapter XV, Riemann's Surfaces. The first pages of this chapter are, however, far from clear. Thus the author speaks (§ 173) of "the theorem that

sheets of a Riemann's surface are joined along lines"; and the form of the remainder of the paragraph is that of a proof of this "theorem". In fact, in the next two paragraphs as well, it is difficult to distinguish between what is intended as definition and what is intended as proof. Cor. I on p. 341 is not correct; consider $w' = 1 - z^4$ and draw the branch-lines as the straight lines -1, 0, $+1$, and $-i$, 0, $+i$. A careful study of some simple examples of conformal representation, illustrated by skilfully designed diagrams, might well precede such general discussion. Examples are, indeed, given in § 176, but they are inadequate. From § 178 on, the presentation is much clearer. The relation between the connectivity $(2p + 1)$, the number of simple branch-points, and the number of sheets is established and the existence of canonical systems of irreducible circuits is made clear; the question of how such a system is to be found in a given case is not considered. Finally, the case of a Riemann's surface with only simple branch-points—a generalization, in a way, of the hyperelliptic surface of two sheets—is discussed, and a canonical set of branch-lines is introduced, § 189. A simple figure would have made the result much more striking. The way in which a canonical system of irreducible circuits is to be taken on such a surface is at once evident.

Chapter XVI, Algebraic Functions and their Integrals, is a brief sketch of some of the most important relations. Here, again, it is to be regretted that the author makes no use of conformal representation, especially as he could thus give some important illustrations of the subjects of the previous two chapters. We mention, in passing, that sufficient restrictions have not been made to justify the conclusion at the end of § 193; the equation $F(w', z) = 0$ may be reducible. The integral of an algebraic function on a Riemann's surface with simple branch-points is considered. The chapter closes with the definition of the Abelian integrals of the three kinds.

The further treatment of algebraic functions is now interrupted by the proof of the existence theorem, Chapter XVII, in which Schwarz's proof is given. The problem to be discussed is stated clearly in the opening paragraphs of the chapter, but the reader who seeks here a rigorous exposition of Schwarz's rigorous analysis will seek in vain; for, as has already been pointed out, rigor of form nowhere marks the author's work and here such deficiency is particularly hampering. In the proof of any complicated theorem in limits, it is of prime importance that it be stated distinctly what magnitudes are chosen arbitrarily and what ones are chosen subsequently to and in a measure dependent on these. This obvious requirement the author does not observe. See, for example, § 219 in this chapter and § 82. Again, the line re-

ferred to in V on p. 422 must not be tangent to the boundary– an essential condition that is not mentioned. But such criticisms as this might be made throughout the book.—The proof given in the second paragraph on p. 416 is not sound. The lemma on p. 410 is a truism as it stands; it should be amended to read: "... and if u_1 and u_2 be the same for any two-dimensional part (no matter how small) of the common portion T, ..." At the close of § 227 the boundary, it is stated, can be made a point. This statement is interesting in connection with Schwarz's words on the last page of the *Auszug aus einem Briefe an Herrn F. Klein, Math. Ann.*, vol. 21, pp. 157–160; *Ges. Werke*, II., p. 305. In this letter, Schwarz indicates briefly how his method can be applied to closed surfaces. A fuller statement of the details would have been acceptable to the reader.

In Chapter XVIII, the study of Abelian integrals is again taken up. Relations between their periods are established and normal integrals are introduced. In § 239, Abelian functions are defined. The reason that the author gives for not defining an Abelian function as the inverse of an Abelian integral is extraordinary. He has shown in § 110 that, if z_0 is any point within the region of continuity of an Abelian integral $w(z)$ and A an arbitrarily assigned real or complex quantity, a path can be so chosen for the integral that $A - w(z_0)$ will be, in absolute value, as small as is desired. (He thinks he has shown that $A - w(z_0)$ can be made to vanish.) Because of these relations he says that z can no longer be considered as a function of w! That is to say, before we can regard Cauchy's or Weierstrass's or Riemann's definition of a function as really defining a function, we must first make sure that such relations as the above do not exist! The real reason for the usual definition of the Abelian functions is, of course, in order to bring the inverse of the integral into the closest possible relation with *single-valued* functions. Now the inverse function is the root of an algebraic equation whose coefficients are single-valued functions, not, to be sure, of a single integral, but of p integrals. These coefficients are defined as Abelian functions.—In closing, the author gives a brief sketch of the allied functions, the integrands of the Abelian integrals. We believe that the author is right in adopting Riemann's methods for the treatment of the algebraic functions and their integrals. For these methods have now been rigorously established, while in the algebraic treatment, which Noether has done so much toward developing, many details still remain to be worked out. But his object is not solely to lay the foundation of a systematic treatment; it is also, in part, to make the student acquainted with some of the simplest properties of these and of their allied functions.

For this purpose, plane curves, rational transformations, and conformal representations are most useful means, and we regret that they have not appealed to the author.—An account of Appell's "factorial functions" and a discussion of the conditions under which the algebraic differential equation $f\left(\frac{dw}{dz}, w\right) = 0$ defines w as a single-valued function of z conclude the chapter.

Chapter XIX, Conformal Representation: Introduction, opens with a discussion of the general problem of the conformal transformation of one surface on another, and several examples are given in illustration. With § 257, the more special problem of the conformal transformation of one plane on another is taken up and a large number of examples, including some of the regular bodies, follow. The sequence of these examples might be amended with advantage, and the whole presentation would be clearer if better illustrated by diagrams. The last twelve pages of the chapter are given to the linear transformation. Its fundamental properties are, indeed, set forth in clear analytic form; but surely the reader may demand that, on a transformation of such importance, the strongest possible geometric light be thrown.

Chapter XX begins with an account of Schwarz's investigations on Riemann's theorem regarding the conformal transformation of a singly connected surface on a circle. Schwarz gave a proof of the theorem under certain very general restrictions. Any account of Schwarz's work that is to be of real value to the student must state accurately the leading ideas on which the successive theorems rest, the chain of necessary conditions that determine the function completely on the supposition that it exist, and finally the argument by which the existence of such a function is established. For, the object of Schwarz's investigations is not to give a heuristic method, but an analytic proof. Now it may be that the author intended a presentation of this sort; but if he did, he has certainly been unfortunate in his style. The conformal transformation on a circle of polygons whose sides are straight lines or arcs of circles is treated and examples are given. In § 269, we read: "For general values of α, β, γ the integral function w is an Abelian transcendent . . . " This is true only when α, β, γ are rational numbers. Next comes a study of the case that the polygon in the w plane is a triangle whose sides are arcs of circles. The corresponding differential equation of the third order and its resolvent linear differential equation of the second order are obtained. Figures are given for the regular bodies and these diagrams illustrate remarkably well the divisions of the sphere. The equations of the regular bodies are deduced directly in the non-homo-

geneous form. The chapter closes with a proof of Schwarz's theorem that "the only equations $\chi(z, w) = 0$ which can give families of algebraical curves in the z plane as the conformal equivalent of the parallel lines, $u =$ constant, in the w plane, are such that z is connected by an algebraical equation either with w, or with a simply periodic function of w, or with a doubly periodic function of w." (p. 579.)

The last two chapters of the treatise are entitled, respectively, Groups of Linear Substitutions, and Automorphic Functions. The first of these takes up rather fully the general properties of such groups, without going into the more subtle questions that arise in connection with them, and it forms a good introduction to the study of these groups. In the last chapter, automorphic functions are defined and illustrated by the simplest examples of such functions, the rational functions belonging to the regular bodies and the elliptic modular functions. Here, it would have been easy for the author to point out the essential distinction between the elliptic functions of Jacobi and Legendre and those of Weierstrass in relation to their invariance as regards the substitutions of the modular group. Poincaré's pseudo-automorphic Θ series are introduced and some properties of automorphic functions, defined as the quotient of such series, are deduced.

It will be seen from the points to which attention has been called that the book is not one that can safely be put into the hands of the immature student for a first introduction to the study of the theory of functions. But the student who is already familiar with its elements and who has acquired some degree of critical power will find its pages incentive to valuable work in this wide field.

W. F. Osgood.

Harvard University, *February*, 1895.

ON THE INTRODUCTION OF THE NOTION OF HYPERBOLIC FUNCTIONS.*

BY PROFESSOR M. W. HASKELL.

THE difficulties in the way of a satisfactory geometrical deduction of the fundamental formulæ of the hyperbolic functions seem to be due to the lack of a definition of these functions which shall be independent of the particular position of the argument area. A general definition of this kind can, however, readily be found in terms of the ratios of certain *areas*, instead of *lines*. From this definition the addition-theorem and other characteristics can be easily deduced by the methods of analytic geometry; and the definitions hold, furthermore, not merely for the rectangular, but for *any* hyperbola.

I. *The circular functions.* In order to bring out clearly the analogy with the circular functions, I will first indicate briefly how the latter would be defined according to this method.

In a circle of radius a (Fig. 1) let ϕ be the angle between the radii OP and OQ, and let OP' be drawn perpendicular to

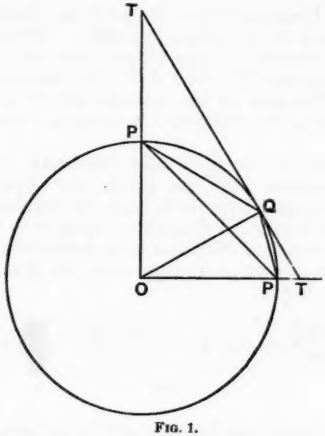

FIG. 1.

*Read before the AMERICAN MATHEMATICAL SOCIETY, December 28, 1894. For various geometrical definitions of these functions, see Professor A. Macfarlane's paper: "On the definition of the trigonometric functions," 1894.—EDITORS.

OP. The following areas are either well known or easily found (the sign Δ denoting triangle):

$$\text{sector } OPQ = \tfrac{1}{2}a^2\phi, \qquad \Delta OPQ = \tfrac{1}{2}a^2 \sin\phi,$$

$$\Delta OQP' = \tfrac{1}{2}a^2 \cos\phi, \quad \Delta OPP' = \tfrac{1}{2}a^2,$$

from which follow immediately:

$$\phi = \frac{\text{sector } OPQ}{\Delta OPP'}, \quad \sin\phi = \frac{\Delta OPQ}{\Delta OPP'}, \quad \cos\phi = \frac{\Delta OQP'}{\Delta OPP'}.$$

Similarly, if the tangent at Q meet OP in T and OP' in T', it is not difficult to show that

$$\tan\phi = \frac{\Delta OTQ}{\Delta OPP'}, \qquad \operatorname{ctn}\phi = \frac{\Delta OQT'}{\Delta OPP'},$$

$$\sec\phi = \frac{\Delta OTP'}{\Delta OPP'}, \qquad \operatorname{csc}\phi = \frac{\Delta OPT'}{\Delta OPP'},$$

Now, these formulæ might be taken as *definitions* of the argument ϕ and of its various functions. They can then be immediately extended to any ellipse, the only modification necessary being that OP and OP' shall be conjugate semi-diameters. The area of the triangle OPP' is then $= \frac{1}{2}ab$, and ϕ is equal to the difference between the eccentric angles P and Q.

II. *Definition of the hyperbolic functions.* Since of two conjugate diameters only one meets the hyperbola in real points, the conjugate hyperbola must be employed also, and P' is the point where the diameter conjugate to OP meets the conjugate hyperbola. We shall then define the argument u and its functions in strict analogy with the preceding results, as follows (see Fig. 2):

$$u = \frac{\text{sector } OPQ}{\Delta OPP'}, \quad \sinh u = \frac{\Delta OPQ}{\Delta OPP'}, \quad \cosh u = \frac{\Delta OQP'}{\Delta OPP'},$$

etc.

If now the hyperbola be referred to its principal axes as axes of coördinates, its equation may be written

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (1)$$

Let x_1, y_1 be the coördinates of P, and x_2, y_2 those of Q. Then the coördinates of P' will be $\frac{ay_1}{b}, \frac{bx_1}{a}$, and the area of the triangle OPP' is equal to $\frac{1}{2}ab$. The definitions of sinh u and cosh u become

$$\sinh u = \frac{x_1y_2 - x_2y_1}{ab}, \quad \cosh u = \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}. \tag{2}$$

Interchanging the coördinates of P and Q, we have

$$\sinh(-u) = -\sinh u, \quad \cosh(-u) = \cosh u. \tag{3}$$

Also, if P and Q coincide, $u = 0$ and

$$\sinh 0 = 0, \quad \cosh 0 = 1. \tag{4}$$

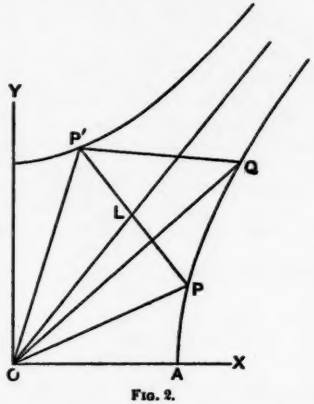

FIG. 2.

Let $u_1 = \frac{\text{sector } OAP}{\triangle OPP'}$ and $u_2 = \frac{\text{sector } OAQ}{\triangle OPP''}$, so that $u = u_2 - u_1$. The coördinates of A being a, 0, we have

$$\sinh u_1 = \frac{y_1}{b}, \ \cosh u_1 = \frac{x_1}{a}; \ \sinh u_2 = \frac{y_2}{b}, \cosh u_2 = \frac{x_2}{a}. \tag{5}$$

Comparing with (1), we see at a glance that

$$\cosh^2 u_1 - \sinh^2 u_1 = 1, \tag{6}$$

and that this result is general may be instantly verified by (2), for

$$\cosh^2 u - \sinh^2 u = \frac{(b^2x_1x_2 - a^2y_1y_2)^2 - a^2b^2(x_1y_2 - x_2y_1)^2}{a^4b^4}$$

$$= \frac{b^2x_1^2 - a^2y_1^2}{a^2b^2} \cdot \frac{b^2x_2^2 - a^2y_2^2}{a^2b^2} = 1.$$

III. *The Addition-Theorem.* Substituting (5) in the definitions of (2), we have

$$\left.\begin{aligned} \sinh(u_2 - u_1) &= \sinh u_2 \cosh u_1 - \cosh u_2 \sinh u_1 \\ \cosh(u_2 - u_1) &= \cosh u_2 \cosh u_1 - \sinh u_2 \sinh u_1 \end{aligned}\right\} \quad (7)$$

Writing $+ u_1$ instead of $- u_1$, we have by (3)

$$\left.\begin{aligned} \sinh(u_2 + u_1) &= \sinh u_2 \cosh u_1 + \cosh u_2 \sinh u_1 \\ \cosh(u_2 + u_1) &= \cosh u_2 \cosh u_1 + \sinh u_2 \sinh u_1 \end{aligned}\right\} \quad (8)$$

The generality of these formulæ is easily verified in the manner just exemplified in formula (6).

IV. It is clear from (5) that the definitions we have given reduce to the ordinary form for the special position there considered. It remains to be shown that in the form given $\sinh u$ and $\cosh u$ are really functions of u alone. To this end let us choose the asymptotes as axes of coördinates. Let α and β be the coördinates of any point, and let ω be the angle between the asymptotes; the equation of the hyperbola is then

$$\alpha\beta = \tfrac{1}{2}ab \csc \omega. \quad (9)$$

The coördinates of P being $\alpha_1 = OL$ and $\beta_1 = LP$, those of Q being α_2 and β_2, those of P' will be α_1 and $-\beta_1$, and we have

$$\sinh u = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{ab \csc \omega}, \quad \cosh u = \frac{\alpha_2\beta_1 + \alpha_1\beta_2}{ab \csc \omega}, \quad (10)$$

or, if we write $\frac{\alpha_2}{\alpha_1} = \lambda$ and therefore $\frac{\beta_2}{\beta_1} = \lambda^{-1}$,

$$\sinh u = \tfrac{1}{2}(\lambda - \lambda^{-1}), \cosh u = \tfrac{1}{2}(\lambda + \lambda^{-1}). \quad (11)$$

If we now apply to the plane the linear transformation

$$\alpha' = k\alpha, \quad \beta' = k^{-1}\beta, \tag{12}$$

the hyperbola is transformed into itself. The points P and Q are moved to any arbitrary position on the curve, but the ratio $\alpha_2 : \alpha_1$ is unaltered. It is easily shown that in this transformation the area of any triangle, and hence the area of any figure in the plane, is unchanged; so that u, $\sinh u$ and $\cosh u$ are unchanged. They are therefore all functions of the ratio $\alpha_2 : \alpha_1$ alone. Hence $\sinh u$ and $\cosh u$ are functions of u alone, and the definitions here given are a proper generalization of the usual definitions.

V. *The exponential formulæ.* The sector OPQ may be regarded as the limit of a circumscribed polygon and hence u may be regarded as the limit of the sum of a series of hyperbolic sines. To make each of the terms in this series equal, we have evidently only to put $\lambda = \rho^n$, where n may be any whole number. Then, writing *lt* briefly for *limit*,

$$u = \tfrac{1}{2}\,\mathrm{lt}\, n(\rho - \rho^{-1}) = \mathrm{lt}\,\frac{n}{2}(\lambda^{\frac{1}{n}} - \lambda^{-\frac{1}{n}}) \tag{13}$$

$$= \mathrm{lt}\,\frac{\lambda^{\frac{2}{n}} - 1}{\frac{2}{n}} \cdot \lambda^{-\frac{1}{n}}, \text{ as } n \text{ increases indefinitely.}$$

The limit of $\lambda^{-\frac{1}{n}}$ is equal to 1, and the limit of $\dfrac{\lambda^{\frac{2}{n}} - 1}{\frac{2}{n}}$ is

the natural logarithm of λ. Hence $\lambda = e^u$. Introducing this value of λ in (12), we have

$$\sinh u = \tfrac{1}{2}(e^u - e^{-u}), \quad \cosh u = \tfrac{1}{2}(e^u + e^{-u}). \tag{14}$$

BERKELEY, CAL., *November* 26, 1894.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held Saturday afternoon, February 23, at three o'clock, the president, Dr. Hill, in the chair. There were fifteen members present. On the recommendation of the council the following persons, nominated at the preceding meeting, were elected to membership: Mr. Joseph Germains Charles Cottier, Jersey City, N. J.; Mr. Alfred Tennyson De Lury, University of Toronto, Toronto; Professor Thomas Franklin Holgate, Northwestern University, Evanston, Ill.; Professor Franklin Pierce Matz, New Windsor College, New Windsor, Md.; Mr. John William Odell, University of Toronto, Toronto; Professor William Fogg Osgood, Harvard University, Cambridge. Three nominations for membership were received. The council announced that a general session of the Society would be held this summer at Springfield, Mass., in connection with the meeting of the American Association for the Advancement of Science. The following paper was presented:

"The n-fold sphere," by Dr. T. PROCTOR HALL.

In the absence of the author the paper was read by the secretary.

THE AMERICAN MATHEMATICAL SOCIETY will hold its second summer meeting at Springfield, Mass. The meeting will begin on Tuesday morning, August 27, at ten o'clock. The general sessions of the American Association for the Advancement of Science, which is to meet in the same place, will begin on the following Thursday morning at the same hour.

THE Sadlerian Professorship of Pure Mathematics at the University of Cambridge, made vacant by Professor Cayley's death, has been filled by the election of Dr. A. R. Forsyth.

SIR JAMES COCKLE, F.R.S., formerly president of the London Mathematical Society, died on January 28, at the age of seventy-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CAPELLI (A.). Lezioni di algebra complementare, ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. Palermo, 1894. 8vo. 526 pp. Fr. 8.00

GRASSMANN (H.). Gesammelte mathematische und physikalische Werke. Auf Veranlassung der K. Sächs. Gesellschaft der Wissenschaften und unter Mitwirkung von J. Lüroth, E. Study, J. Grassmann, H. Grassmann d. J. und G. Scheffers herausgegeben von F. Engel. (In 3 volumes.) Band I: Theil 2: Die Ausdehnungslehre von 1862. Leipzig, Teubner, 1895. 8vo. Illustrated. Mk. 16.00

HALUSCHKA (F.). Zur Kegelschnittlehre. [Progr.] Wien, 1894. 8vo. 24 pp. Mk. 1.50

KUMMER (R.). Die Flächen mit unendlich vielen Erzeugungen durch Translation von Kurven. [Diss.] Leipzig, 1894. 8vo. 65 pp. Mk. 1.80

MCCLINTOCK (E.). Theorems in the calculus of enlargement. [*American Journal of Mathematics*, Vol. 17.] 4to. 12 pp.

—— A method for calculating simultaneously all the roots of an equation. [*American Journal of Mathematics*, Vol. 17.] 4to. 22 pp.

MANDL (M.). Ueber einen Satz aus der Theorie der biquadratischen Reste. [Progr.] Prossnitz, 1894. 8vo. 4 pp.

MUTH (P.). Grundlagen für die geometrische Anwendung der Invariantentheorie. Mit einem Begleitworte von M. Pasch. Leipzig, Teubner, 1895. 8vo. vi and 131 pp. Mk. 3.00

SCHLESINGER (L.). Handbuch der Theorie der linearen Differentialgleichungen. (In 2 volumes.) Band I. Leipzig, Teubner, 1895. 8vo. xx and 486 pp. Mk. 16.00

STURM (C.). Cours d'analyse de l'École Polytechnique, revu et corrigé par E. Prouhet et augmenté de la théorie élémentaire des fonctions elliptiques. 10e édition, par A. de St. Germain. 2 vols. Paris, Gauthier-Villars, 1895. 8vo. Fr. 15.00

TANNERY (J.). Leçons d'arithmétique théorique et pratique. Paris, 1894. 8vo. 12 and 510 pp.

TIKHOMANDRITSKY (M.). Elements of the theory of Abelian integrals. [In Russian.] Published by the Kharkov Mathematical Society. Kharkov, Droeder, 1895. 8vo. 16 and 234 pp. Rubles 4.00

VEGAS Y PUEBLA-COLLADO (M.). Tratado de geometria analitica. Madrid, 1894. 4to. 8 and 576 pp. Mk. 18.00

WANGERIN (A.). Ueber die Abwickelung von Flächen constanten Krümmungsmasses sowie einiger andrer Flächen auf einander. [Festschrift.] Halle, 1894. 4to. 22 pp. Mk. 1.20

II. ELEMENTARY MATHEMATICS.

BLACKIE'S first-stage mathematics. Euclid and algebra. London, Blackie, 1895. 8vo. 6d.

BRADSHAW (J. G.). A first step in Euclid. Consisting of the first twelve propositions, with explanations, illustrations and numerous examples. London and New York, Macmillan, 1895. 16mo. 85 pp. Cloth. $0.50

BUKA (F.). Grundzüge der darstellenden Geometrie. Charlottenburg, 1894. 4to. 24 pp. Mk. 1.20

C. (F.). Géométrie des écoles primaires, avec 600 exercices gradués, suivie d'un cours de métrage pratique, comprenant l'arpentage, le levé des plans, le calcul des distances et des hauteurs, le nivellement, et une méthode générale pour le cubage des solides. Rennes, Simon, 1894. 16mo. 199 pp. Illustrated.

——— Exercices de géométrie. Énoncés et solutions des exercices proposés dans la Géométrie des écoles primaires. Rennes, Simon, 1894. 16mo. 242 pp. Illustrated.

DEIGHTON (H.). The elements of Euclid. Books 1 to 4. 3d edition. London, Bell, 1895. 8vo. 3s.

——— The elements of Euclid. Books 5 and 6, and part of book 9. London, Bell, 1895. 8vo. 2s. 6d.

GUÉRIN (A.). L'allievo d'Archimede. Lezioni di geometria. 260 figure. Milano, Legato, 1895. Fr. 1.25

JUBINAL (H.). Démonstration du postulatum d'Euclide. Tarbes, Croharé, 1894. 8vo. 31 pp. 3 plates.

KLEYER (A.) und SACHS (J.). Lehrbuch der ebenen Elementar-Geometrie (Planimetrie). Nach System Kleyer. Theil VII: Die Aehnlichkeit der geradlinigen Figuren, von J. Sachs. Stuttgart, Maier, 1894. 8vo. 8 and 165 pp. Mk. 4.00

LENER (P.). Die Mathematik und ihre Axiome. Bamberg, 1894. 12mo. 36 pp. Mk. 0.20

MITTENZWEY (L.). Mathematische Kurzweil oder 333 Aufgaben, Kunststücke, geistanregende Spiele, Ueberraschungen, verfängliche Schlüsse, Scherze und dgl. aus der Zahlen- und Formenlehre für jung und alt zur Unterhaltung und Belehrung. 3te Auflage. Leipzig, Klinkhardt, 1895. 8vo. 120 pp. Mk. 1.50

ORIO Y RUBIO (M.). Libro de problemas aritmeticos, algebraicos y geometricos. Parte 1. Madrid, 1894. 4to. Mk. 6.00

SANNIA E D'OVIDIO. Elementi di geometria. 9a edizione. Napoli, 1894. 8vo. 661 pp. Illustrated. Fr. 6.50

SCHRÖN (L.). Logarithmen. Tafel I und II: 7-stellige gemeine Logarithmen der Zahlen von 1 bis 108,000 und der Sinus, Cosinus, Tangenten und Cotangenten aller Winkel des Quadranten von 10 zu 10 Secunden. 22te Ausgabe. Braunschweig, Vieweg, 1894. 8vo. 8 and 498 pp. Mk. 4.20

SERRET (J. A.). Trattato di trigonometria piana e sferica. Traduzione da F. Grassi, con note ed aggiunte. 3a edizione. Firenze, 1894. 12mo. Fr. 4.00

SÖDERBLOM (A.). Descriptiv Geometri. Göteborg, 1894. 8vo. 50 pp. Mk. 4.00

III. APPLIED MATHEMATICS.

ABNEY (W. DE W.). Colour vision: being the Tyndall lectures delivered in 1894 at the Royal Institution. London, Low, 1895. 8vo. 232 pp. 1 colored plate. 12s. 6d.

AUWERS (A.). Tobias Mayer's Sternverzeichniss nach den Beobachtungen auf der Göttinger Sternwarte in den Jahren 1756–1760, neu bearbeitet. Leipzig, Engelmann, 1894. 4to. 7 and 208 pp. Mk. 22.00

BAUER (L. A.). Beiträge zur Kenntniss des Wesens der Säcular-Variation des Erdmagnetismus. [Diss.] Berlin, Mayer & Müller, 1895. 4to. 56 pp. Mk. 3.00

BEZOLD (W. VON). Hermann von Helmholtz. Gedächtnissrede. Mit einem Portrait nach einem Oelgemälde von Frz. von Lenbach. Leipzig, Barth. 8vo. 31 pp. Mk. 1.50

BRISSE (C.) et ANDRÉ (C.). Cours de physique. 3me édition par C. Brisse et C. Rivière. Paris, 1894. 8vo. 752 pp. Illustrated. Fr. 12.50

CASTELLANO. Lezioni di meccanica razionale. Torino, 1894. 8vo. 512 pp. Fr. 12.00

DRUDE (P.). Die Theorie in der Physik. [Antrittsvorlesung.] Leipzig, Hirzel. 8vo. 15 pp. Mk. 0.80

DUHEM (P.) Commentaire aux principes de la thermodynamique. [*Journal de math. pures et appl.*, 4e série, Vol. X (1894), pp. 207–285.]

FORTSCHRITTE DER PHYSIK im Jahre 1888. Dargestellt von der physikalischen Gesellschaft zu Berlin. 44ter Jahrgang. 3te Abtheilung. Braunschweig, Vieweg, 1894. 8vo. Mk. 30.00

GIAXA (V. DE). Riflessioni sui vari metodi proposti per la determinazione della latitudine col mezzo delle altezze circummeridiane. [Progr.] Lusinpiccolo, 1894. 8vo. 43 pp.

GUILHAUMON (J. B.). Résumé de la théorie du navire. Paris, 1894. 8vo. 1 plate. Mk. 2.30

HAMBURGER (F.). Über die Elektrizitätsleitung in schlechtleitenden Flüssigkeiten. [Diss.] Leipzig, 1894. 8vo. 65 pp.

HOPPE (O.). Elementares Lehrbuch der technischen Mechanik für Studirende und zum Selbstunterricht. 2te Abtheilung: Mechanik der tropfbaren und gasförmigen Flüssigkeiten. Leipzig, Felix, 1895. 8vo. xi and 135 pp. Illustrated. Mk. 4.50

JORDAN (W.). Mathematische und geodätische Hülfstafeln. 9te Aufl. Hannover, Helwing, 1895. 8vo. 120 pp. Mk. 1.00

LARKIN (H.). Elliptical orbits; their distinctive mechanical characteristics and their possible origin. London, Unwin, 1895. 8vo. 1s.

LOMMEL (E. VON). Lehrbuch der Experimentalphysik. 2te Auflage. Leipzig, 1894. 8vo. 12 and 550 pp. Illustrated. Mk. 6.40

LORENTZ (H. A.). Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern. Leiden Brill, 1895. 8vo. iii and 139 pp. Mk. 2.50

MACH (E.). Popular scientific lectures. Translated by T. J. McCormack. Chicago, Open Court Pub. Co., 1895. 8vo. 5 and 313 pp. Illustrated. Cloth. $1.00

MIDDLETON (G. A. T.). Stresses and thrusts; a text-book for students; being an enlarged edition of "Strains in Structures." London, Batsford, 1895. 8vo. 150 pp. Illustrated. 5s.

MORAWETZ (J.). Einige Bewegungen unveränderlicher Systeme. [Progr.] Salzburg, 1894. 8vo. 28 pp.

NICHOLS (E. L.). Laboratory manual of physics and applied electricity. With the co-operation of E. Merritt, F. J. Rogers and others Volume II: Senior course and outline of advanced work. 1894. 8vo. Cloth. $4.00

PROCTOR (R. A.) and RANYARD (A. C.). Old and new astronomy. London, Longmans, 1895. Folio. 824 pp. With plates. 21s.

SILBERSTEIN (L.). Ueber die mechanische Auffassung elektromagnetischer Erscheinungen in Isolatoren und Halbleitern. [Diss.] Berlin, 1894. 8vo. 42 pp.

THOMPSON (S. P.). Elementary lessons in electricity and magnetism. New revised edition with additions. N. Y., Macmillan, 1895. 8vo. 15 and 286 pp. Illustrated. Cloth. $1.40

TURNER (A.). Die Kraft und Materie im Raume. Grundlage einer neuen Schöpfungstheorie. 4te Auflage. Leipzig, 1894. 8vo. 21 and 378 pp. Mk. 12.00

WISLICENUS (W. F.). Astronomische Chronologie Ein Hilfsbuch für Historiker, Archäologen und Astronomen Leipzig, 1895. 8vo. Cloth. Mk. 4.00

WITTSACK (P.). Einführung in die Festigkeitslehre. Hildburghausen, Kesselring, 1895. 8vo. viii and 152 pp. Mk. 4 00